# Test of Mathematics for University Admission, 2020 Paper 2 Worked Solutions

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## Contents

## Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be 'definitive' solutions.

The questions themselves are available on the 'Preparing for the test' section on the Admissions Testing website.

## Question 1

We find the points where the line and curve meet by solving the equations simultaneously; this gives $x^2 + kx + 2 = x - 2$. Rearranging gives

$$x^2 + (k-1)x + 4 = 0.$$

The discriminant of this is $(k-1)^2 - 4 \times 1 \times 4 = (k-1)^2 - 16$. The line and curve cross or touch when the discriminant is $\geq 0$, so we must solve

$$(k-1)^2 - 16 \geq 0.$$

Expanding and simplifying this gives

$$k^2 - 2k - 15 \geq 0,$$

which we can factorise to get

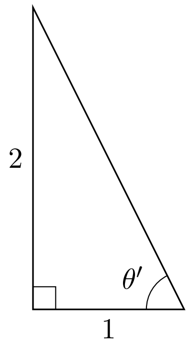
$$(k-5)(k+3) \geq 0.$$

The equation $(k-5)(k+3) = 0$ has roots $k = -3$ and $k = 5$, so the quadratic is non-negative when $k \leq -3$ or $k \geq 5$. Thus the correct solution is E.

## Question 2

In the range $180° < \theta < 360°$, we have that $\tan\theta > 0$ when $180° < \theta < 270°$. In this smaller range, $\cos\theta < 0$.

Next, in the range $0° < \theta' < 90°$, if $\tan\theta' = 2$, we can sketch a right-angled triangle with this angle:




The hypotenuse has length $\sqrt{2^2 + 1^2} = \sqrt{5}$, so $\cos\theta' = \dfrac{1}{\sqrt{5}} = \dfrac{\sqrt{5}}{5}$.

Therefore for our original angle $\theta$, we have $\cos\theta = -\dfrac{\sqrt{5}}{5}$, which is option F.

## Question 3

Lines (I) to (V) are algebraic manipulations; we need to check whether each manipulation is correct.

Line (I) writes $4 = 2 \times 2$, which is correct.

Line (II) multiplies out the inner brackets, but does not do it correctly: the inner brackets should expand to $(9n + 1) - (3n - 1)$, which simplifies to $9n + 1 - 3n + 1$.

The remaining manipulations are correct.

Thus the first error occurs on line (II), and the correct option is C.

## Question 4

A counterexample is a positive integer $N$ that is greater than 6 but it is not the case that $N$ can be written as the sum of two non-prime integers that are greater than 1.

- $N = 5$ is not greater than 6, so is not a counterexample.
- $N = 7 = 2 + 5 = 3 + 4$ cannot be written in this way, so this provides a counterexample.
- $N = 9 = 2 + 7 = 3 + 6 = 4 + 5$ also cannot be written in this way.

Therefore both II and III provide counterexamples; the correct option is G.

## Question 5

Note that none of these graphs have scales on.

A key question is what happens to $y$ when $x$ is large and positive and when $x$ is large and negative.

If $x = 100$, say, then $y = \dfrac{2^{100}}{1+2^{100}}$. Since $2^{100}$ is a very large positive number, this fraction is almost exactly 1. This excludes graphs B and C, where the value of $y$ increases towards infinity. It also excludes graphs D and E, where the value of $y$ tends towards 0. We are therefore left with graphs A and F.

One way to distinguish between A and F is to compare the value at $x = 0$ with the value when $x$ is large. When $x = 0$, we have $y = \frac{1}{2}$. Since $\frac{1}{2} < 1$ and $y$ is close to 1 when $x$ is large, graph F is not possible, so the answer is graph A.

Alternatively, we can consider what happens when $x = -100$, say. Then $y = \dfrac{2^{-100}}{1+2^{-100}}$. Since $2^{-100}$ is a tiny positive number, very close to 0, this fraction is almost exactly 0. This excludes all of the graphs except for A and B; combining this with our earlier observations shows that the correct graph is A.

## Question 6

If we think about the integrals as meaning 'the (signed) area under the graph of $y = \mathrm{f}(x)$ between $x = -5$ and $x = 0$' and the same between $x = 0$ and $x = 5$, we realise that none of these conditions is necessary (though the first two conditions are actually sufficient).

To show fully that a condition is not necessary, we need to find an example of a function for which the integral equation is satisfied but for which the condition is not.

An example of a function which satisfies the integral equation but does not satisfy conditions II or III is the function $\mathrm{f}(x) = x^2$.

An example of a function which satisfies the integral condition but does not satisfy condition I is $\mathrm{f}(x) = sin2\pi x$, which has 5 full periods in each of the ranges $-5 \le x \le 0$ and $0 \le x \le 5$, so each of the integrals is equal to 0, but it is not the case that $\mathrm{f}(x) = \mathrm{f}(-x)$ for $-5 \le x \le 5$.

Therefore the correct option is A.

## Question 7

To show that a condition is sufficient, we must explain why having the condition means that the parallelogram must be a square. To show that a condition is not sufficient, we must find an example of a parallelogram that satisfies the condition but is not a square.

For condition I, if $PQRS$ is a rhombus that is not a square, then it satisfies the condition but is not a square. So condition I is not sufficient.

For condition II, the same example works: the diagonals of a rhombus intersect at right angles even when the rhombus is not a square.

For condition III, we know that the adjacent angles $\angle PQR$ and $\angle QRS$ sum to $180°$, so if condition III is satisfied, they must each be $90°$, a right angle. But a rectangle that is not a square satisfies this condition and is a parallelogram.

Therefore none of these conditions is sufficient, and the correct option is H.

## Question 8

To prove that a 'for all' statement is true, it is necessary to show that the formula holds for every choice of the variables. On the other hand, to show that it is false, it is sufficient to just find a single counterexample, because that shows that it is not true for *all* choices of the variables.

This student has provided a correct counterexample (and it is easy to check that their arithmetic is correct). Therefore the student has correctly shown that $(*)$ is false, and the correct option is E.

One can also observe that 'for all real numbers $a$ and $b$, $|a+b| \leq |a| + |b|$' is a true statement, but noting that is different from showing that $(*)$ is false.

## Question 9

The student wishes to evaluate f(4). Writing 4 rad to explicitly indicate radians, this becomes

$$\mathrm{f}(4) = 4\sin(4\,\mathrm{rad}) = 4 \times \sin(4\,\mathrm{rad}).$$

Now

$$\pi\,\mathrm{rad} = 180°,$$

so dividing by $\pi$ gives

$$1\,\mathrm{rad} = \frac{180°}{\pi}.$$

Multiplying this by 4 then gives

$$4\,\mathrm{rad} = \frac{4 \times 180°}{\pi}.$$

Therefore the student could type into their calculator

$$4 \times \sin(4 \times 180 \div \pi)$$

which is equivalent to the calculation of F.

## Question 10

One result about inequalities is that if $p < q$ and $r < s$, then $p + r < q + s$. This is because

$$p + r < q + r < q + s$$

where the first inequality follows by adding $r$ to $p < q$ and the second follows by adding $q$ to $r < s$.

Since we have $a + b < c + d$ and $a + c < b + d$, we can therefore add these to get

$$2a + b + c < b + c + 2d.$$

Subtracting $b + c$ from both sides gives $2a < 2d$, so $a < d$. Hence I must be true.

A similar approach does not seem to work to show $b < c$, which suggests that perhaps we should look for a counterexample. If $d$ is much larger than $a$, then the values of $b$ and $c$ should not make much of a difference to the inequality if they are both small. So let us try $a = 1$, $d = 100$, $b = 2$, $c = 1$. Then we do indeed have $0 < 1 + 2 < 1 + 100$ and $0 < 1 + 1 < 2 + 100$, so the first two inequalities hold but $b > c$. Therefore II might not be true.

For III, we note that $a + b > 0$ and $c + d > a + b > 0$, so adding $a + b > 0$ and $c + d > 0$ gives $a + b + c + d > 0$, and thus III must be true.

The correct option is F.

## Question 11

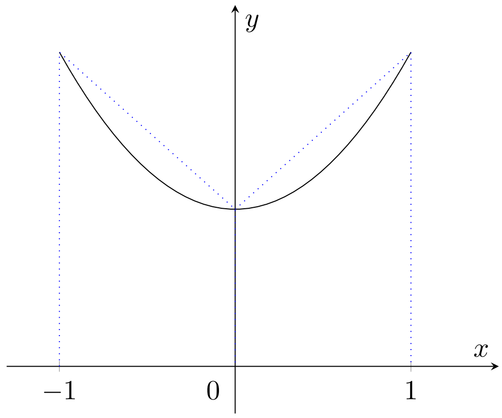
From the nature of the pattern, we see that it sort-of repeats every 2 units along each axis. So let us simplify this question by replacing 99 and 100 by 3 and 4 (an odd number and the even number one more than it).

Looking at the portion of the graph we are given, we see the points $(3, 4)$ and $(4, 3)$ are both on the spiral (top right), $(3, -4)$ and $(4, -3)$ are too (bottom right), $(-4, -3)$ and $(-3, -4)$ are both on the bottom left and if we continue the line at the top, $(-3, 4)$ is at the top left. However $(-4, 3)$ is missed out: that is the gap at the end of one 'loop' as the spiral heads out to form the next 'loop'.

Therefore in the case of 99 and 100, $(-100, 99)$ will be missed, which is option G.

## Question 12

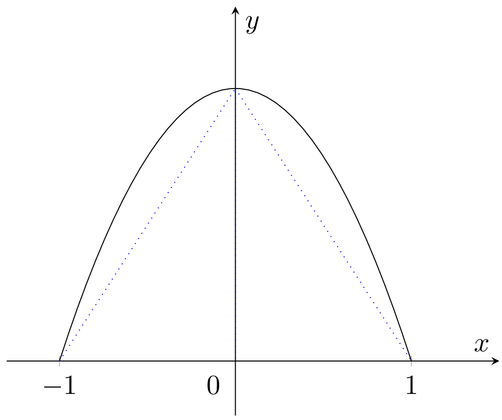
The trapezium rule can produce an overestimate even if the function's gradient goes from negative to positive, for example with the function $f(x) = x^2 + 1$ between $x = -1$ and $x = 1$:




Therefore neither the condition $f'(x) > 0$ for all $x$ with $a < x < b$ nor the condition $f'(x) < 0$ for all $x$ with $a < x < b$ is necessary for the trapezium rule to produce an overestimate. This rejects the 'only if' options (B, C, E, F), leaving us with only A and D.

The above quadratic can also be used to help distinguish between these two options. We have $f'(x) = 2x$ and $f''(x) = 2$. Thus for $-1 < x < 0$, $f'(x) < 0$ and $f''(x) > 0$ and the trapezium rule (with one strip) gives an overestimate. So D appears plausible.

On the other hand, we could reflect the quadratic to change the sign of $f''(x)$: consider the function $f(x) = 1 - x^2$.




We now have $f'(x) = -2x$ and $f''(x) = -2$. So when $-1 < x < 0$, we have $f'(x) > 0$ and $f''(x) < 0$, yet the trapezium rule (with one strip) produces an underestimate. This shows that A cannot be correct.

By elimination, the correct option must be D.

It is also possible to justify this: a function with $f''(x) > 0$ is 'bending upwards' (the technical term is *convex*). Any time we draw a chord between two points on the graph, the chord will lie above the graph (except at the endpoints), so the trapezium rule estimate will be an overestimate of the integral. In fact, this argument makes no assumptions about the sign of either $f(x)$ or of $f'(x)$; having $f''(x) > 0$ for $a < x < b$ is sufficient for the trapezium rule to give an overestimate (but it is not necessary).

## Question 13

This question is about inequalities. To begin with, the only thing that we know for certain is that $\big(\mathrm{f}(x)\big)^2 \geq 0$ for all $x$, as it is the square of a real number. It follows that $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x \geq \int_0^3 \mathrm{f}(x)\,\mathrm{d}x$, as we are adding a non-negative number to $\int_0^3 \mathrm{f}(x)\,\mathrm{d}x$ to get $\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$. Therefore II is necessarily true.

Let us now take II and rewrite the integral from 0 to 3 as the sum of the integrals from 0 to 1 and from 1 to 3; this is so that we can match the integral on the right hand side. We obtain:

$$\int_0^1 \mathrm{f}(x)\,\mathrm{d}x + \int_1^3 \mathrm{f}(x)\,\mathrm{d}x \leq \int_0^1 \mathrm{f}(x)\,\mathrm{d}x$$

and we can now subtract the integral from 0 to 1 to get

$$\int_1^3 \mathrm{f}(x)\,\mathrm{d}x \leq 0.$$

Since the integral is negative or zero, the function itself must be so somewhere between 1 and 3; if it were positive throughout the interval, the integral would be positive. Therefore condition I must also be true.

Hence the correct answer is option D.

## Question 14

*Approach 1: direct reasoning*

We can see from the example exactly what is needed for a sequence $T$ to have property P: the sum of the terms from the $(m+1)$th term to the $(2m)$th term must be zero.

Consider now the two statements.

I We will have $ad < 0$ if $a > 0$ and $d < 0$. But if $|d|$ is large enough, then the second term will be negative and there is no hope of a sum of terms giving zero. For example, the sequence $1, -2, -5, \ldots$ has $ad = 1(-2) = -2 < 0$, but does not have property P. So this statement is false.

II In the given example, $d$ is even and $m$ is also even. But if $m$ is odd, we would need the middle term of the second half to be zero, and then it does not matter whether $d$ is even or odd, for example in the sequence (and this was constructed by working backwards): 4, 3, 2, 1, 0, $-1$, ..., which has property P taking $m = 3$. An even more extreme example is the sequence 1, 0, ..., where $d = -1$ again and $m = 1$. Therefore this statement is false.

Hence the correct option is A.

*Approach 2: algebraic approach*

The sum of the first $m$ and the first $2m$ terms of the sequence are given by

$$\begin{aligned} S_m &= \tfrac{1}{2}m(2a + (m-1)d) \\ S_{2m} &= \tfrac{1}{2}(2m)(2a + (2m-1)d) \end{aligned}$$

and these are equal if and only if

$$\tfrac{1}{2}m(2a + (m-1)d) = \tfrac{1}{2}(2m)(2a + (2m-1)d).$$

Dividing both sides by $\frac{1}{2}m$ gives

$$2a + (m-1)d = 2(2a + (2m-1)d).$$

Expanding the brackets and rearranging gives

$$2a + (3m-1)d = 0. \tag{$*$}$$

I The equation $(*)$ can be rearranged to make $m$ the subject, giving

$$m = \frac{1}{3}\left(-\frac{2a}{d} + 1\right).$$

There is no reason why this should be an integer whenever $ad < 0$. For example, taking $a = 1$ and $d = -5$ gives $m = \frac{1}{3} \times \frac{7}{5}$ which is not an integer. Therefore this statement is false.

II We can rearrange $(*)$ to make $d$ the subject:

$$d = -\frac{2a}{3m-1}$$

but it is actually more helpful to rearrange it as

$$3m - 1 = -\frac{2a}{d}\,.$$

To show that the statement is true, we need to show that whenever we have integer values for $a$, $d$ and $m$, $d$ will be even; to show that the statement is false, we need to find an example where $d$ is odd. If we take $d = 1$, we need a value of $a$ for which $-2a$ is one less than a (positive) multiple of 3; $-2a = 2$ and $-2a = 8$ both work, giving $a = -1$ and $a = -4$ respectively. Similarly, if we take $d = 3$, we need a value of $a$ for which $-\frac{2a}{3}$ is one less than a (positive) multiple of 3; here, $-2a = 2 \times 3 = 6$ would work, giving $a = -3$.

Any one of these counterexamples would show that this statement is false.

Again, we see that the correct option is A.

## Question 15

It is very helpful to write out the values of the summands (the terms being added) in terms of $\sqrt{3}$ and similar, and then work out the first few values of the sum.

| $k$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | $\dots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\frac{k\pi}{3}$ | $\frac{\pi}{3}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ | $\frac{5\pi}{3}$ | $2\pi$ | $\frac{7\pi}{3}$ | $\frac{8\pi}{3}$ | $3\pi$ | $\frac{10\pi}{3}$ | $\dots$ |
| $\sin(\frac{k\pi}{3})$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{2}$ | $0$ | $-\frac{\sqrt{3}}{2}$ | $-\frac{\sqrt{3}}{2}$ | $0$ | $\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{2}$ | $0$ | $-\frac{\sqrt{3}}{2}$ | $\dots$ |
| $\sum_{k=1}^{n} \sin(\frac{k\pi}{3})$ | $\frac{\sqrt{3}}{2}$ | $2\frac{\sqrt{3}}{2}$ | $2\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{2}$ | $0$ | $0$ | $\frac{\sqrt{3}}{2}$ | $2\frac{\sqrt{3}}{2}$ | $2\frac{\sqrt{3}}{2}$ | $\frac{\sqrt{3}}{2}$ | $\dots$ |

We can see that the sum repeats every 6 steps, and that it equals $\frac{\sqrt{3}}{2}$ twice in every cycle of 6, specifically when $n = 1, 4, 7, 10, \dots$. This sequence of $n$ values are the numbers which are 1 more than a multiple of 3, and so the correct option is D.

## Question 16

We can check whether the calculation stands any chance of being correct by checking whether line (V) is correct. We can work out the integral directly:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_x^{2x} t^2\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left[\tfrac{1}{3}t^3\right]_x^{2x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}(2x)^3 - \tfrac{1}{3}x^3\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{7}{3}x^3\right)\\ &= 7x^2\end{aligned}$$

So an error has certainly been introduced somewhere.

From this, it is clear that line (IV) is also incorrect.

Line (I) is correct, as $\int_a^b \mathrm{f}(t)\,\mathrm{d}t + \int_b^c \mathrm{f}(t)\,\mathrm{d}t = \int_a^c \mathrm{f}(t)\,\mathrm{d}t$ for all functions $\mathrm{f}(t)$ and real numbers $a$, $b$ and $c$.

Therefore the error must lie in line (II) or (III). We can work out the left hand sides of the equations explicitly as we did for line (V):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{x} t^2\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left[\tfrac{1}{3}t^3\right]_0^{x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{1}{3}x^3\right)\\ &= x^2\end{aligned}$$

so line (II) is correct. However,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{2x} t^2\,\mathrm{d}t\right) &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\left[\tfrac{1}{3}t^3\right]_0^{2x}\right)\\ &= \frac{\mathrm{d}}{\mathrm{d}x}\left(\tfrac{8}{3}x^3\right)\\ &= 8x^2\end{aligned}$$

so line (III) is incorrect. Hence the correct option is D.

The reason for the error is that the fundamental theorem of calculus is stated for the case where the upper limit is $x$; if the upper limit is $2x$, then the integral grows twice as fast as $x$ increases, so we actually have $\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_0^{2x} t^2\,\mathrm{d}t\right) = 2(2x)^2 = 8x^2$.

## Question 17

Let us write the six integers as two sets of three, each set in increasing order:

$$a,\ 8,\ b \qquad c,\ 9,\ d$$

We know the middle integers are 8 and 9 because these are the medians.

The first set of three has a mean of 10 and so they must sum to $3 \times 10 = 30$, therefore $a+b = 22$. Likewise, the second set sums to 36 and so $c + d = 27$.

We also know that $a \leq 8 \leq b$ and $c \leq 9 \leq d$.

Here are tables of some of the possibilities for $a$ and $b$ and for $c$ and $d$; the two tables are independent of each other:

| $a$ | $b$ |
|---|---|
| 8 | 14 |
| 7 | 15 |
| 6 | 16 |
| 5 | 17 |
| 4 | 18 |
| 3 | 19 |

| $c$ | $d$ |
|---|---|
| 9 | 18 |
| 8 | 19 |
| 7 | 20 |
| 6 | 21 |
| 5 | 22 |
| 4 | 23 |

The range of the set of all six integers is the larger of $b$ and $d$ minus the smaller of $a$ and $c$.

As all six integers are distinct, we cannot have $a = 8$ or $c = 9$ or $c = 8$. We also want $b$ and $d$ to be as small as possible while $a$ and $c$ are as large as possible to get a small range.

Let us try $a = 7$, $b = 15$. Then we cannot have $c = 7$, so we try $c = 6$, $d = 21$. This gives a range of $21 - 6 = 15$.

If we now try $a = 6$, $b = 16$, we can take $c = 7$, $d = 20$, and this gives a range of $20 - 6 = 14$, which is smaller.

If we next try $a = 5$, $b = 17$, we can again take $c = 7$, $d = 20$, but now the range has grown to $20 - 5 = 15$ again. And if we go smaller with the choice of $a$, say $a = 4$, $b = 18$, then the smallest range we can obtain becomes even larger: $20 - 4 = 16$. The range will keep growing as we take smaller values for $a$.

So the smallest possible range is 14, which is option E.

## Question 18

Let us write $\mathrm{f}(x) - \mathrm{g}(x)$ explicitly:

$$\mathrm{f}(x)-\mathrm{g}(x) = (ax^3+bx^2+cx+d)-(px^3+qx^2+rx+s) = (a-p)x^3+(b-q)x^2+(c-r)x+(d-s).$$

We are given that this cubic is positive for every real $x$. But a cubic always crosses the $x$-axis! The only way to reconcile this is if the cubic $\mathrm{f}(x)-\mathrm{g}(x)$ is not actually a cubic, because $a-p=0$.

So we have $a-p=0$ and statement I is false. We can also now write

$$\mathrm{f}(x) - \mathrm{g}(x) = (b-q)x^2 + (c-r)x + (d-s).$$

If this function is always positive, then there are two possibilities: either $b-q$ is nonzero and the quadratic has a negative discriminant, or $b-q=0$ and this is not actually a quadratic. The second option looks quite like statement II: if $b-q=0$, that is, $b=q$, then we have

$$\mathrm{f}(x) - \mathrm{g}(x) = (c-r)x + (d-s).$$

Since a linear function always crosses the $x$-axis, this cannot be always positive unless $c-r=0$, in which case the function is actually just the constant $\mathrm{f}(x) - \mathrm{g}(x) = (d-s)$. (To be more precise, this is still a linear function, just with zero $x$ coefficient.) Therefore, if $\mathrm{f}(x) - \mathrm{g}(x) > 0$ for all real $x$, it follows that if $b=q$, then $c=r$, and so statement II is true.

We have reached the end of this line of reasoning, and we have not stumbled upon anything that looks like statement III. But $d$ and $s$ are the constant coefficients in the polynomials f and g, so if we take $x=0$, we have $\mathrm{f}(0)=d$ and $\mathrm{g}(0)=s$. Since $\mathrm{f}(0)-\mathrm{g}(0)>0$, this shows that $d-s>0$, or $d>s$, hence statement III is true.

The correct option is G.

## Question 19

The person in the centre gives us a lot of information. If they are a truth-teller, then all four neighbours are liars, but if they are a liar, then at least one neighbour is a truth-teller. Since we are only interested in the number of truth-tellers and not their positions, we can assume the person directly 'above' them is a truth-teller. So we have these two possibilities (where T means truth-teller and L means liar):

| | | |
|---|---|---|
| | L | |
| L | T | L |
| | L | |

| | | |
|---|---|---|
| | T | |
| | L | |
| | | |

In the first possibility, each corner position has two neighbours, both of whom are liars, so they must be occupied by truth-tellers. In the second possibility, the top two corners must be liars, giving these possibilities:

| | | |
|---|---|---|
| T | L | T |
| L | T | L |
| T | L | T |

Case 1: 5 T, 4 L

| | | |
|---|---|---|
| L | T | L |
| | L | |
| | | |

In the right-hand possibility, the corner liars already have one neighbour who is a truth-teller, so the other neighbour could be either a truth-teller or a liar. This gives four possibilities for the left side and right side people: both liars, both truth-tellers or one of each in either order. By reflecting the grid, we only need to consider one case of one liar, one truth-teller, so we now have these three possible ways of continuing the grid:

| | | |
|---|---|---|
| L | T | L |
| L | L | L |
| | | |

| | | |
|---|---|---|
| L | T | L |
| T | L | T |
| | | |

| | | |
|---|---|---|
| L | T | L |
| L | L | T |
| | | |

Every truth-teller is surrounded by liars, and every liar must have at least one truth-teller as a neighbour, so we can fill in some of the bottom row in each of these three possibilities:

| | | |
|---|---|---|
| L | T | L |
| L | L | L |
| T | | T |

| | | |
|---|---|---|
| L | T | L |
| T | L | T |
| L | | L |

| | | |
|---|---|---|
| L | T | L |
| L | L | T |
| T | | L |

We can now fill in the final square in each of these, and check that the three people in the bottom row are consistent with the rules:

| | | |
|---|---|---|
| L | T | L |
| L | L | L |
| T | L | T |

Case 2: 3 T, 6 L

| | | |
|---|---|---|
| L | T | L |
| T | L | T |
| L | T | L |

Case 3: 4 T, 5 L

| | | |
|---|---|---|
| L | T | L |
| L | L | T |
| T | L | L |

Case 4: 3 T, 6 L

We therefore have four cases. The smallest number of truth-tellers is in cases 2 and 4, with 3 truth-tellers, and the largest number is in case 1 with 5 truth-tellers. Therefore the correct option is E.

## Question 20

It is helpful to write all six statements as 'if ... then' statements:

A **if** $x \geq 0$ **then** $\mathrm{f}(x) < 0$

B **if** $\mathrm{f}(x) \geq 0$ **then** $x < 0$

C **if** $x \geq 0$ **then** $\mathrm{f}(x) \geq 0$

D **if** $x < 0$ **then** $\mathrm{f}(x) < 0$

E **if** $\mathrm{f}(x) \geq 0$ **then** $x \geq 0$

F **if** $x < 0$ **then** $\mathrm{f}(x) \geq 0$ **and if** $\mathrm{f}(x) \geq 0$ **then** $x < 0$

If F is true, then B is also true, so F is false.

Let us suppose that A is true. Then its contrapositive is also true. The contrapositive of A is: **if** $\mathrm{f}(x) \geq 0$ **then** $x < 0$, which is statement B. So A and B are either both true or both false. Since exactly one statement is true, A and B must both be false.

Similarly, D and E are contrapositives of each other, so they must both be false as well.

The only statement left is C, so that must be true.

Furthermore, C contradicts A, so A (and B) must be false; C is independent of D (knowing that C is true does not give any information about whether D is true or false), and C contradicts F, so F must be false.

Therefore having C true and the other five statements false is consistent; the function $\mathrm{f}(x) = |x|$ is an example of a function for which C is true and the other five statements are false.

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