Test of Mathematics for University Admission (TMUA)
2023 Paper 2 Worked Solutions

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Introduction for students

These solutions are designed to support you as you prepare to take the Test of Mathematics for University Admission. They are intended to help you understand how to answer the questions, and therefore you are strongly encouraged to **attempt the questions first** before looking at these worked solutions. For this reason, each solution starts on a new page, so that you can avoid looking ahead.

The solutions contain much more detail and explanation than you would need to write in the test itself – after all, the test is multiple choice, so no written solutions are needed, and you may be very fluent at some of the steps spelled out here. Nevertheless, doing too much in your head might lead to making unnecessary mistakes, so a healthy balance is a good target!

There may be alternative ways to correctly answer these questions; these are not meant to be 'definitive' solutions.

The questions themselves are available on the TMUA Preparation Materials

Question 1

We first multiply both sides of the equation by $11(\sqrt{x}-6)(\sqrt{x}+6)$ to eliminate all of the fractions, giving

$$11\left(\sqrt{x}+6\right)-11\left(\sqrt{x}-6\right)=3\left(\sqrt{x}-6\right)\left(\sqrt{x}+6\right).$$

We can now expand all of the brackets to give

$$11\sqrt{x}+66-11\sqrt{x}+66=3x-108,$$

which simplifies to

$$240=3x.$$

Dividing by 3 gives $x=80$, which is option H.

Question 2

We first expand the brackets to obtain

$$\int_9^{16} \frac{1}{x} + 2 + x \,\mathrm{d}x - \int_9^{16} \frac{1}{x} - 2 + x \,\mathrm{d}x.$$

Since these integrals have the same limits, we can combine them into a single integral to get

$$\int_9^{16} \left(\frac{1}{x} + 2 + x\right) - \left(\frac{1}{x} - 2 + x\right) = \int_9^{16} 4 \,\mathrm{d}x = 4 \times (16 - 9) = 28,$$

which is option F.

Note that integration of $\frac{1}{x}$ *is not on the specification for this test; it is, of course, possible to answer this question by calculating both integrals explicitly and then subtracting, but it is much more work than the solution just presented.*

Question 3

We substitute the values into the two sides of the equation:

I $\sqrt{1^{16}} = \sqrt{1} = 1$ and $1^{\sqrt{16}} = 1^4 = 1$, so this is not a counterexample.

II $\sqrt{2^8} = 2^4 = 16$ and $2^{\sqrt{8}} \neq 16$, so this is a counterexample.

III $\sqrt{3^4} = 3^2 = 9$ and $3^{\sqrt{4}} = 3^2 = 9$, so this is not a counterexample.

Therefore the answer is option C, II only.

Question 4

The argument looks very convincing and there is no obvious error. However, writing down the first few odd integers and underlining those which are prime shows that there is a problem:

$$1, \underline{3}, \underline{5}, \underline{7}, 9, \underline{11}, \underline{13}, 15, \underline{17}, \underline{19}, \ldots$$

There are three consecutive odd numbers that are all prime right at the start (3, 5, 7), so the student's final answer in line VII is incorrect. We must therefore find the first error in the argument.

I This line is correct: 17 and 19 are consecutive odd integers that are prime.

II This is correct: consecutive odd integers differ by 2, and we can call the middle one n.

III If $n = 3k+1$, then $n+2 = 3k+3 = 3(k+1)$, which is a multiple of 3, so this line is correct.

IV If $n = 3k+2$, then $n-2 = 3k$, which is a multiple of 3, so this line is correct.

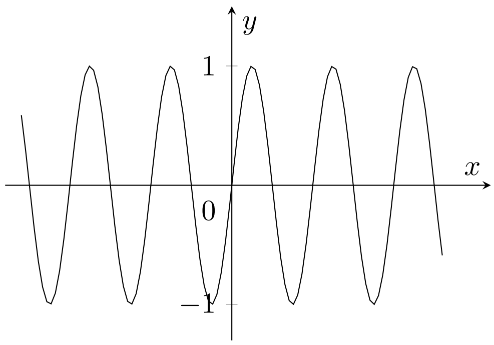
V All integers are either multiples of 3, one more than a multiple of 3 or two more than a multiple of 3; a number which is three more than a multiple of 3 is itself a multiple of 3. Similarly, a number which is four more than a multiple of 3 is one more than the next multiple of 3 (in algebraic notation: $3k+4 = 3(k+1)+1$), and so on. So this statement is correct.

VI In each case, one of $n+2$, $n-2$ and n is a multiple of 3. But our example of 3, 5, 7 shows that the last part of this statement is false: 3 is a multiple of 3, but *is* prime.

So the first error is on line VI, which is option G.

Question 5

The graph of $y = \sin 2x$ is periodic with period π: it is the graph of $y = \sin x$ stretched by a factor of $\frac{1}{2}$ in the x-direction:



So if R is true, we see from the graph that $\int_0^k \sin 2x \,\mathrm{d}x = 0$ as the areas above and below the x-axis cancel. Thus R is sufficient for S.

Conversely, if $\int_0^k \sin 2x \,\mathrm{d}x = 0$, the areas above and below the x-axis must cancel. We can see from the graph that the only way to achieve this is if k is a multiple of π, so that we have an exact number of periods of $y = \sin 2x$. (The integral increases as k increases from 0 to $\frac{\pi}{2}$, and then decreases back to 0 as k increases from $\frac{\pi}{2}$ to π, and the same happens in every period.) Therefore S implies R, or R is necessary for S.

It follows that the correct answer is option A: R is necessary and sufficient for S.

Question 6

I If we take the logarithm to base a of the equation $(*)$, we get $x = \log_a x$.

This suggests that this equation must have the same number of real solutions as $(*)$, but we must be careful and check that we are allowed to take the logarithm of both sides of $(*)$ without adding or losing solutions. We can only take the logarithm of a positive number. The left hand side of $(*)$ is always positive, and if $x \leq 0$, then this value of x does not give a solution to $(*)$. So the solutions to $(*)$ (if there are any) all have $x > 0$. So we can take the logarithm of both sides without losing any solutions. And by exponentiating the resulting equation, we get back to $(*)$, so we do not introduce any new solutions. Therefore this equation must have the same number of real solutions as $(*)$.

II If we square both sides of $(*)$, we obtain $a^{2x} = x^2$. Squaring can introduce new solutions, though; does it on this occasion? If we have a solution to $a^{2x} = x^2$, we can deduce that $a^x = \pm x$. We always have $a^x > 0$, so it might be that we get a new solution with $x < 0$. Let us try a numerically simple example to see if this does happen. As $x < 0$, we have $0 < a^x < 1$, so we need $-1 < x < 0$. Taking $x = -\frac{1}{2}$, $(*)$ becomes $a^{-1/2} = \frac{1}{2}$; we then get $a = 4$. So in this case, $4^{2x} = x^2$ has an extra solution, $x = -\frac{1}{2}$, which is not a solution to $4^x = x$. Therefore this equation may not have the same number of real solutions as $(*)$.

Another way to think about it is graphically. The graphs of $y = a^x$ and $y = a^{2x}$ are both exponential graphs with $y < 1$ for all $x < 0$. The graph of $y = x$ is negative for all $x < 0$ so any solutions to $(*)$ have $x > 0$. On the other hand, $y = x^2$ is positive for all $x \neq 0$ and it reaches 1 at $x = -1$. Therefore there will always be a solution to $a^{2x} = x^2$ with $-1 < x < 0$.

III If we replace $2x$ with u in this equation, we get $a^u = u$, so there are the same number of real solutions for u as $(*)$ has. Every (real) value of u corresponds to a unique value of x, namely $x = u/2$, so there must be the same number of real solutions to $a^{2x} = 2x$ as $(*)$.

Thus the answer is option F, I and III only.

Question 7

We can rearrange the equation to give $by = -ax + c$, so

$$y = -\frac{a}{b}x + \frac{c}{b}.$$

Therefore the gradient is $-\frac{a}{b}$ and the y-intercept is $\frac{c}{b}$.

We go through the conditions to determine whether they are necessary and/or sufficient for a positive gradient and positive y-intercept.

A $\frac{c}{b} > 0$ is equivalent to the y-intercept being positive. Next, $\frac{a}{b} < 0$ is equivalent to $-\frac{a}{b} > 0$, which is equivalent to the gradient being positive. So this is a necessary and sufficient condition.

B $\frac{c}{b} < 0$ is equivalent to the y-intercept being negative, so this is neither necessary nor sufficient.

C If we take $a > 0 > c > b$, then $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, so the gradient and y-intercept are both positive. Therefore $a > b > c$ is not necessary.

D If we take $a < 0 < c < b$, then $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, so the gradient and y-intercept are both positive. Therefore $a < b < c$ is not necessary.

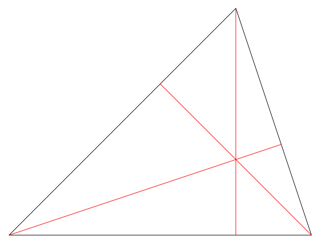
E If $-\frac{a}{b} > 0$ and $\frac{c}{b} > 0$, then $\frac{a}{b} < 0 < \frac{c}{b}$, so a and c must have opposite signs. Therefore this condition is necessary. But if a and c have opposite signs, we know nothing about the sign of b, so we might or might not have $-\frac{a}{b} > 0$. So this condition is not sufficient.

F As in E, if the gradient and y-intercept are both positive, a and c must have opposite signs, so this condition is not necessary.

The correct option is therefore E.

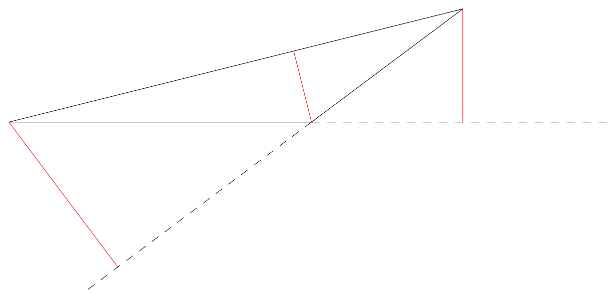
Question 8

Let us start with a triangle and see how many such straight lines it has. If we take an acute-angled triangle, then it has three such lines (shown in red), which are the altitudes of the triangle, going from each vertex perpendicular to the opposite side:

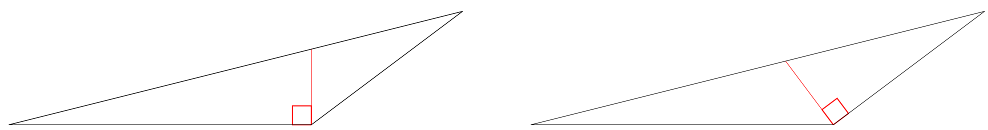


Therefore statement III is true.

But can the student draw a triangle for which there are fewer than 3 such straight lines? There is always such a straight line (altitude) from every vertex in an acute-angled triangle, so the only hope for fewer than 3 such straight lines is to consider an obtuse-angled triangle. There is only one usable altitude in this case, and that is from the obtuse angle; the other two altitudes land outside the triangle:



So it may appear that there is only one such line in this case. However, we are not restricted to altitudes! Instead, we can split the obtuse angle into a right angle and an acute angle in two different ways, giving another two lines:



Neither of these lines can meet the side opposite the obtuse angle perpendicularly, as if they did, we would have a triangle with two right angles, so they are always distinct from the altitude.

So in this case, we also get three such staight lines.

Hence the correct answer is option D, III only.

Question 9

We first note that the interior angles in a pentagon sum to $3 \times 180° = 540°$ (since a pentagon can be cut up into three triangles). Therefore the mean interior angle is $540°/5 = 108°$.

I The angles could be $108°$, $108°$, $108°$, $107°$ and $109°$: start with a regular pentagon and slightly rotate one of the sides. These do not form an arithmetic sequence, so statement $(*)$ is false.

II The contrapositive of a statement always has the same truth value as the statement itself, so this is also false.

III The converse of $(*)$ states: **if** all the interior angles in P form an arithmetic sequence, **then** at least one of the interior angles in P is $108°$. If the interior angles in P form an arithmetic sequence with common difference d, then the angles can be written as $c-2d$, $c-d$, c, $c+d$, $c+2d$. (One could also write them as a, $a+d$, $\ldots$, $a+4d$, but this approach is easier.) Then the sum of the interior angles is $5c = 540°$, so $c = 108°$. Since one of the interior angles is c, the converse of $(*)$ is true.

Thus the answer is option D, III only.

Question 10

Substituting $x = 0$ gives $x^4 - 2x^2 - 3 = -3 < 0$, so the final conclusion on line VI is at least plausible. We will check the argument line-by-line.

I This is obtained by adding 4 to both sides, and this is reversible. This line is correct.

II This is an algebraic rearrangement: $(x^2 - 1)^2 = x^4 - 2x^2 + 1$ is an identity, so this line is correct.

III We have $u^2 < 4$ if and only if $-2 < u < 2$. Taking $u = x^2 - 1$, we see that this line is equivalent to line II, so this step is correct.

IV This looks suspicious, as one of the inequalities has been dropped. The statement $x^2 - 1 < 2$ certainly follows from $-2 < x^2 - 1 < 2$, but the claim is that they follow from each other. Can we deduce $-2 < x^2 - 1 < 2$ from $x^2 - 1 < 2$?

Perhaps surprisingly, it turns out that we can. The function $x^2 - 1$ has a minimum value of -1, so we always have $x^2 - 1 \geq -1$, and therefore $-2 < x^2 - 1$ is true for all x. Hence the statement in line III does follow from $x^2 - 1 < 2$, and this line is correct.

V This is obtained by adding 1 to both sides, which is reversible, so this line is correct.

VI Similarly to our observation in line III, $x^2 < 3$ is true if and only if $-\sqrt{3} < x < \sqrt{3}$, so this line is correct.

Thus the entire argument is correct, and the correct option is A.

Question 11

I This is the converse of $(*)$, so is not equivalent to $(*)$.

II We can rewrite this as an 'if… then' statement, noting that '**A** only if **B**' means the same as 'if **A**, then **B**'. So this becomes '**if** 2^k+1 is **not** prime, **then** k is **not** a power of 2'. This is the contrapositive of '**if** k is a power of 2, **then** 2^k+1 is prime', which is the statement in I, namely the converse of $(*)$. So this statement is equivalent to the converse of $(*)$ but not equivalent to $(*)$ itself. (The contrapositive of the converse, which is the same as the converse of the contrapositive, is called the *inverse* of the original statement.)

III '**A** is a sufficient condition for **B**' is the same as 'if **A**, then **B**', so this statement can be rewritten as '**if** 2^k+1 is prime, **then** k is a power of 2'. (Note the subtlety of language here: the condition **A** is '2^k+1 is prime' and **B** is 'k is a power of 2', but they are written in the order 'a sufficient condition for **B** is **A**'.) This is equivalent to $(*)$ (and is in fact just a different way of saying the same thing).

The correct option is G.

> ***Commentary:*** *It turns out that statement I is false. It was conjectured by Pierre de Fermat in the 17th century, who noted that* $2^{2^n}+1$ *is prime for* $n=1$*, 2, 3 and 4. Such primes are called Fermat primes. The smallest counterexample to statement I is* $2^{2^5}+1=641\times 6\,700\,417$*, which was discovered by Leonhard Euler in the 18th century. It is not known whether* $2^{2^n}+1$ *is prime for any* $n>4$*.*

Question 12

We can rewrite the two statements in if/then language as

I if $n = 3$, then $p > 1$

II if $n = 7$, then $-1 < p < 1$

We will determine whether these statements are true or false using this if/then formulation.

The equation can be rearranged as $\sin x(\cos^2 x - p^2) = 0$, so this is true if and only if $\sin x = 0$ or $\cos x = \pm p$. The first of these has solutions $x = 0, \pi, 2\pi$, while the solutions to the second depend on the value of p:

- If $p = 0$, then it has solutions $x = \frac{\pi}{2}, \frac{3\pi}{2}$.
- If $-1 < p < 0$ or $0 < p < 1$, then it has four solutions, one with $0 < x < \frac{\pi}{2}$, one with $\frac{\pi}{2} < x < \pi$, one with $\pi < x < \frac{3\pi}{2}$ and one with $\frac{3\pi}{2} < x < 2\pi$; these are all different from the solutions to $\sin x = 0$.
- If $p = \pm 1$, then it has three solutions, $x = 0, \pi, 2\pi$, which are exactly the same as the solutions of $\sin x = 0$.
- If $p < -1$ or $p > 1$, then it has no solutions.

Therefore the original equation has:

- $n = 5$ distinct solutions if (and only if) $p = 0$
- $n = 7$ distinct solutions if (and only if) $-1 < p < 0$ or $0 < p < 1$
- $n = 3$ distinct solutions if (and only if) $p = \pm 1$ or $p < -1$ or $p > 1$

We can now return to the statements I and II in our rewritten form:

I This is false: we could have $p = 1$ or $p \leq -1$

II This is true, even though $p = 0$ is included here: if $-1 < p < 0$ or $0 < p < 1$, then certainly $-1 < p < 1$ is true

The correct option is therefore C, II only.

Question 13

We work through them sequentially:

A If $x \geq 0$, then B may be false, C may be false, D may be false and E may be false

B If $x = 1$, then A is true, C is true, D is true and E is true

C If $x = 0$ or $x = 1$, then A is true, B may be false, D is true and E is true

D If $x \geq 0$ or $x \leq 1$, then A may be false (for example if $x = -1$), B may be false, C may be false and E may be false

E If $x \geq 0$ and $x \leq 1$, then A is true, B may be false, C may be false and D is true

The correct option is therefore C, which is sufficient for exactly three of the other four statements.

Question 14

We rewrite these equations in the form $y = mx + c$:

$$y = -\frac{a}{b}x - \frac{c}{b}$$
$$y = -\frac{b}{c}x - \frac{a}{c}$$
$$y = -\frac{c}{a}x - \frac{b}{a}$$

Since these equations are cyclically symmetrical (replacing $a \to b \to c \to a$ leaves the set of equations unchanged), we may assume the two parallel or perpendicular lines are the first two.

A If the first two lines are parallel, then $\frac{a}{b} = \frac{b}{c}$, but this does not imply that $\frac{c}{a}$ is equal to these. For example we could have $a = 1$, $b = 2$, $c = 4$ giving $\frac{a}{b} = \frac{b}{c} = \frac{1}{2}$ but $\frac{c}{a} = 4$.

B The same example shows that the third line need not be perpendicular to the other two. In fact, in this case, since $\frac{a}{c} = \left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right) = \left(-\frac{a}{b}\right)^2 > 0$, the third line will always have a negative gradient (namely $-\frac{c}{a}$), even if the first two also have negative gradients.

C The same example shows that this is not necessarily the case; in fact, as the third line always has a negative gradient, it cannot be parallel to $y = x$.

D Again, the same example shows that the third line might not have gradient -1.

E If the first two lines are perpendicular, we have $\left(-\frac{a}{b}\right)\left(-\frac{b}{c}\right) = \frac{a}{c} = -1$, so $-\frac{c}{a} = 1$. It is not obvious why the lines should meet at a point, so let us return to this after we have looked at the remaining options.

F This is true: the calculation in E shows that the third line has gradient 1.

G This is false: the third line has gradient 1, not -1.

So the correct option is F.

We still need to eliminate option E. Let us consider an example with $a = 1$, $b = 2$; we therefore need $c = -1$ as $-\frac{c}{a} = 1$. Thus the three lines are

$$y = -\frac{1}{2}x - \frac{-1}{2}$$
$$y = -\frac{2}{-1}x - \frac{1}{-1}$$
$$y = -\frac{-1}{1}x - \frac{2}{1}$$

which simplifies to

$$y = -\tfrac{1}{2}x + \tfrac{1}{2}$$
$$y = 2x + 1$$
$$y = x - 2$$

The second and third lines meet at $(-3, -5)$, but the first line passes through $(-3, 2)$, so these three lines do not all meet at a point.

Question 15

Approach 1

Using the rule for working out the value of a base 2 number, we have

$$\begin{aligned}0.\dot{0}01\dot{1} &= 0.001100110011\ldots\\ &= 0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4}+0\times 2^{-5}+0\times 2^{-6}+1\times 2^{-7}+1\times 2^{-8}+\cdots\end{aligned}$$

Grouping these into sets of four consecutive terms gives

$$\begin{aligned}0.\dot{0}01\dot{1} &= (0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4})+\\ &\quad (0\times 2^{-5}+0\times 2^{-6}+1\times 2^{-7}+1\times 2^{-8})+\cdots\\ &= (0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4})+\\ &\quad 2^{-4}(0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4})+2^{-8}(\cdots)+\cdots\\ &= \left(\frac{1}{8}+\frac{1}{16}\right)+2^{-4}\left(\frac{1}{8}+\frac{1}{16}\right)+2^{-8}\left(\frac{1}{8}+\frac{1}{16}\right)+\cdots\\ &= \frac{3}{16}(1+2^{-4}+2^{-8}+\cdots)\end{aligned}$$

The term in brackets is an infinite geometric series with first term 1 and common ratio 2^{-4}, so its sum is

$$\frac{1}{1-2^{-4}}=\frac{1}{1-\frac{1}{16}}=\frac{16}{15}.$$

Therefore

$$0.\dot{0}01\dot{1}=\frac{3}{16}\times\frac{16}{15}=\frac{3}{15}=\frac{1}{5}$$

which is option B.

Approach 2

We have, on dropping zero terms and rearranging:

$$\begin{aligned}0.\dot{0}01\dot{1} &= 0.001100110011\ldots\\ &= 0\times 2^{-1}+0\times 2^{-2}+1\times 2^{-3}+1\times 2^{-4}+0\times 2^{-5}+0\times 2^{-6}+1\times 2^{-7}+1\times 2^{-8}+\cdots\\ &= 2^{-3}+2^{-4}+2^{-7}+2^{-8}+2^{-11}+2^{-12}+\cdots\\ &= (2^{-3}+2^{-7}+2^{-11}+\cdots)+(2^{-4}+2^{-8}+2^{-12}+\cdots)\end{aligned}$$

(Technical note: we are allowed to rearrange this infinite series as all of the terms are positive.)

We can now calculate the sums of these infinite geometric series to get

$$\begin{aligned}0.\dot{0}01\dot{1} &= \frac{2^{-3}}{1-2^{-4}}+\frac{2^{-4}}{1-2^{-4}}\\ &= \frac{2}{16-1}+\frac{1}{16-1}\\ &= \frac{3}{15}\\ &= \frac{1}{5}\end{aligned}$$

Approach 3

We can adapt the method used to evaluate a base 10 recurring decimal.

Let $x = 0.\dot{0}01\dot{1} = 0.001100110011\ldots$. Then writing a subscript '2' to indicate base 2 and a subscript '10' to indicate the usual base 10, we have

$$\begin{aligned} & & x &= (0.001100110011\ldots)_2 \\ &\text{so} & 16_{10}x &= (11.001100110011\ldots)_2 \\ &\text{giving} & 15_{10}x &= 11_2 \end{aligned}$$

Now $11_2 = 1 \times 2^1 + 1 \times 2^0 = 3_{10}$, so we have (in base 10) $15x = 3$, yielding $x = \frac{1}{5}$, which is option B.

Question 16

I Since the highest common factor of a and b is 7, they are both multiples of 7. So u_1 and u_2 are multiples of 7, hence $u_3 = u_1 + u_2$ is a multiple of 7. Then $u_4 = u_2 + u_3$ is a multiple of 7, and so on. So every u_n is a multiple of 7 and this statement is true.

II We have $u_3 = u_1 + u_2 = u_1(1 + u_2/u_1)$. As u_1 is not a factor of u_2, the expression in brackets is not an integer, so u_1 is not a factor of u_3.

Next, we have $u_4 = u_2 + u_3 = u_1 + 2u_2 = u_1(1 + 2u_2/u_1)$. It is feasible that u_1 is a factor of u_4: it will be if $2u_2/u_1$ is an integer. We could take $u_1 = 14$ and $u_2 = 7$ to achieve this. (Explicitly, in this case we have $u_3 = 21$, $u_4 = 28$, and 14 is a factor of 28.)

So this statement may be false.

III Following on from our calculations in II, we have $u_5 = u_3 + u_4 = (u_1 + u_2) + (u_1 + 2u_2) = 2u_1 + 3u_2$. The highest common factor of u_1 and u_5 is therefore the same as the highest common factor of u_1 and $3u_2$, and it is possible that this is 21 if u_1 is a multiple of 21 but u_2 is not. Let us check this explicitly: take $u_1 = 21$ and $u_2 = 7$, so the highest common factor of u_1 and u_2 is 7, as required. Then $u_5 = 2 \times 21 + 3 \times 7 = 3 \times 21$, and so the highest common factor of $u_1 = 21$ and $u_5 = 3 \times 21$ is 21. Therefore this statement may be false.

The correct answer is therefore option B, I only.

Question 17

The value of $2^{\lceil x \rceil}$ changes at every integer. We have

$$2^{\lceil x \rceil} = \begin{cases} 2^0 & \text{when } x = 0 \\ 2^1 & \text{when } 0 < x \leq 1 \\ 2^2 & \text{when } 1 < x \leq 2 \\ 2^3 & \text{when } 2 < x \leq 3 \\ \dots & \\ 2^{99} & \text{when } 98 < x \leq 99 \end{cases}$$

We therefore break the integral up into unit intervals. The value of $2^{\lceil x \rceil}$ at the endpoints does not influence the value of the integral, so we have

$$\begin{aligned} \int_0^{99} 2^{\lceil x \rceil}\,\mathrm{d}x &= \int_0^1 2^{\lceil x \rceil}\,\mathrm{d}x + \int_1^2 2^{\lceil x \rceil}\,\mathrm{d}x + \int_2^3 2^{\lceil x \rceil}\,\mathrm{d}x + \dots + \int_{98}^{99} 2^{\lceil x \rceil}\,\mathrm{d}x \\ &= \int_0^1 2^1\,\mathrm{d}x + \int_1^2 2^2\,\mathrm{d}x + \int_2^3 2^3\,\mathrm{d}x + \dots + \int_{98}^{99} 2^{99}\,\mathrm{d}x \\ &= 2^1 + 2^2 + 2^3 + \dots + 2^{99} \end{aligned}$$

as the integral of a constant is just the constant times the interval width. We now have a geometric series with first term $a = 2$, common ratio $r = 2$ and $n = 99$ terms, so the sum is

$$\frac{a(r^n - 1)}{r - 1} = \frac{2^1(2^{99} - 1)}{2 - 1} = 2^{100} - 2$$

which is option F.

Question 18

Writing $u = x^2$, the equation becomes $u^2 + bu + c = 0$. This has two distinct real roots u_1 and u_2 if and only if $b^2 - 4c > 0$, or $b^2 > 4c$, so this is a necessary condition.

The roots of the original equation are then given by $x = \pm\sqrt{u_1}$ and $x = \pm\sqrt{u_2}$. So for the original equation to have four distinct real roots, we require u_1 and u_2 to be distinct *positive* real numbers. We have

$$u_1, u_2 = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

so the smaller of the two is $\frac{1}{2}\left(-b - \sqrt{b^2 - 4c}\right)$. If $b > 0$, then this will be negative, so we require $b < 0$. Supposing $b < 0$, we also require $-b > \sqrt{b^2 - 4c}$, which becomes on squaring (as both sides are positive) $b^2 > b^2 - 4c$, so $c > 0$.

Putting these all together, the original equation will have four distinct real roots if and only if all of the following hold:

- $b^2 > 4c$
- $b < 0$
- $c > 0$

As $c > 0$, we can deduce that $b^2 > 4c$ if and only if $b > 2\sqrt{c}$ or $b < -2\sqrt{c}$. Since we require $b < 0$, the first is not possible, so the pair of conditions $b < 0$ and $b^2 > 4c$ is equivalent to the single condition $b < -2\sqrt{c}$.

Therefore a necessary and sufficient condition for the original equation to have four distinct real roots is $c > 0$ and $b < -2\sqrt{c}$, which is option D.

Question 19

Since $f(x) = 0$ for exactly M real values of x, $f(x)$ has at least $M - 1$ stationary points: there must be at least one stationary point between every adjacent pair of values of x for which $f(x) = 0$. But there may also be more stationary points between these pairs of values or before the first or after the last value.

Now $g(x) = xf'(x)$ is zero whenever $f(x)$ has a stationary point. So there are at least $M - 1$ values of x for which $g(x) = 0$, but there will be more if $f(x)$ has more than $M - 1$ stationary points. In addition, $g(x) = 0$ when $x = 0$. If $x = 0$ is a stationary point of $f(x)$, then $g(x) = 0$ at every stationary point of $f(x)$ and nowhere else. If $x = 0$ is not a stationary point of $f(x)$, then $g(x) = 0$ will be true for one further value of x.

Summarising:

- the number of stationary points of $f(x)$ is at least $M - 1$
- N equals the number of stationary points of $f(x)$ or one more than the number of stationary points of $f(x)$.

Therefore N is at least $M - 1$, so $N \geq M - 1$, which rearranges to $M \leq N + 1$.

Now considering the options:

I $M < N$ is possible if $f(x)$ has extra stationary points between zeros. For example, $f(x) = x^2 + 1$ has $M = 0$ but $N = 1$. As a second example, let $f(x)$ be a cubic with both turning points above the x-axis; this has $M = 1$ and $N = 2$ or $N = 3$.

II $M = N$ is possible if the number of stationary points of $f(x)$ is $M - 1$ and $x = 0$ is not a stationary point of $f(x)$. For example, $f(x) = (x - 1)^2 - 1$ has $M = 2$ (at $x = 0$ and $x = 2$) and one stationary point at $x = 1$. So $g(x) = 0$ at $x = 0$ and $x = 1$, giving $N = 2$.

III $M > N$ is possible if the number of stationary points of $f(x)$ is $M - 1$ and $x = 0$ is a stationary point of $f(x)$. For example, $f(x) = x^2 - 1$ has $M = 2$ (at $x = \pm 1$) and one stationary point at $x = 0$. So $g(x) = 0$ at $x = 0$ only, giving $N = 1$.

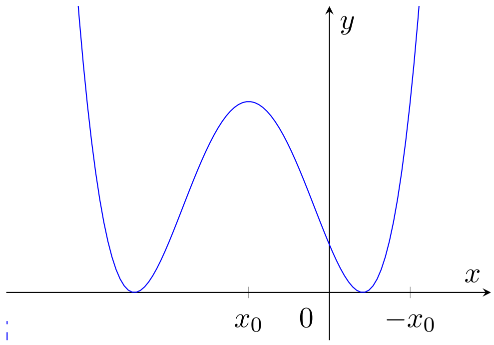
The correct answer is option H, I, II and III.

Question 20

We note first that when $x \geq 0$, $f(|x|) = f(x)$, and so for $x \geq 0$, the integrand is just $\big(f(x)\big)^2 - \big(f(x)\big)^2 = 0$. When $x < 0$, though, the integrand could take any value.

1 We present a complicated approach first.

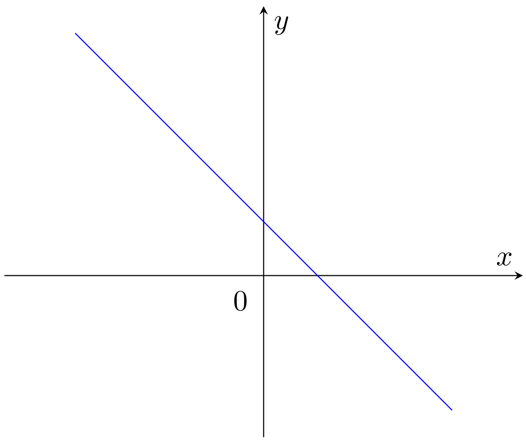
There seems to be no obvious reason why this must be true; one could imagine a polynomial for which the values of the integrand are positive for some negative values of x and negative for others, so the positive parts and negative parts cancel out making the integral zero even though $p < 0$. Here is one example: consider a quartic $f(x)$ that looks roughly like this:



For negative values of x close to $x = 0$, $\big(f(x)\big)^2 > \big(f(|x|)\big)^2$. But once $x < x_0$, we have $\big(f(x)\big)^2 < \big(f(|x|)\big)^2$ as $f(|x|) > f(x)$. We can therefore take $q = 0$ and find some $p < x_0$ for which $I_{p,q} = 0$.

A much simpler approach is that we can take $p = 0$, $q > 0$ and then $I_{p,q} = 0$, so this statement is not true. Were the question to have said $0 \leq p$ instead of $0 < p$, though, we would not have been able to use this simpler approach.

2 We could take $f(x) = 1 - x$, with $f'(x) = -1 < 0$ for all x.



Then $\big(f(x)\big)^2 > \big(f(|x|)\big)^2$ for all $x < 0$, so $I_{p,q} > 0$ for all $p < q < 0$.

3 We observed at the start of this question that the integrand is zero for all $x \geq 0$. So if $p \geq 0$, then $I_{p,q} = \int_p^q 0 \, \mathrm{d}x = 0$. The contrapositive of this statement is: if $I_{p,q} \neq 0$, then $p < 0$. The given statement ($I_{p,q} > 0$ only if $p < 0$, which is the same as saying: if $I_{p,q} > 0$, then $p < 0$) is a special case of the more general result and is therefore true.

The correct answer is therefore option D, 3 only.